Uniqueness of Completions for Linear Time Varying Differential Algebraic Equations

Stephen L. Campbell*
Department of Mathematics
and Center for Research in Scientific Computation
North Carolia State University
Raleigh, North Carolina 27695-8205

Submitted by Stephen Barnett

ABSTRACT

The extent to which a completion $x' = G(t)x + \sum_{i=0}^{r} R_i(t)f^{(i)}(t)$ of a linear time varying differential algebraic equation $E(t)x'(t) + F(t)x(t) = f(t)$ is unique is carefully examined. The implications for numerical methods for solving DAEs based on differentiated equations are discussed.

1. INTRODUCTION

Various techniques have been developed for generating an explicit ordinary differential equation

$$x' = G(t)x + \sum_{i=0}^{r} R_i(t)f^{(i)}(t) \tag{1}$$

from a time varying differential algebraic equation (DAE),

$$E(t)x'(t) + F(t)x(t) = f(t) \tag{2}$$

by differentiation of the equation (2) and solving for x'. Here E is assumed

*Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-87-0051, the U.S. Army Research Office under DAALO3-89-D-0003, and the National Science Foundation under ECS-9012909.

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$3.50

always singular. The system (2) is sometimes called a descriptor system. Applications of this approach have ranged from numerical methods [1–4, 6] to systems inversion, prescribed path control, observability, and other control concepts [6, 8, 11, 12]. More recently this approach has been applied to boundary value problems [9].

In [7] it is pointed out that G and R_i need not be unique even for linear time invariant systems. There is some confusion in the current literature, which we have contributed to, concerning to what extent the G and R_i in (1) are unique. This issue is crucial for considering such questions as the smoothness of the G, R_i and the robustness of the proposed numerical methods. In this note we shall carefully develop, for the first time, the uniqueness and variability of the coefficients in (1). These results have implications for both numerical methods for solving DAEs and control theoretic algorithms. This note examines linear time varying systems, but the issues addressed are also important for nonlinear problems [6].

Section 2 will develop our basic notation and carefully describe how (1) is computed from (2). Section 3 contains the main results of this note.

2. NOTATION AND DEFINITIONS

The DAE (2) is said to be *solvable* on the connected interval $\mathscr{I}$ if for every sufficiently smooth f on $\mathscr{I}$, there is a smooth solution defined on all of $\mathscr{I}$. In addition all solutions are defined on all of $\mathscr{I}$, and solutions are uniquely determined by their value at any $t_0 \in \mathscr{I}$ [5]. This definition of solvability does not require E to have constant rank, nor for it to be possible to carry out the usual inversion algorithms involving coordinate changes and differentiations [5, 12]. To avoid technical problems dealing with various degrees of smoothness we shall assume that E, F, f in (2) are infinitely differentiable. However, less smoothness would suffice.

The system of algebraic equations

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \tag{3}$$

is 1-*full* [2, 3] with respect to x_1 if (3) uniquely determines x_1 for any consistent b. From basic linear algebra we know that 1-fullness is equivalent to the row echelon form of A being

$$\begin{bmatrix} I_{n\times n} & 0 \\ 0 & * \end{bmatrix}$$

where $*$ is a possibly nonzero entry and $n = \dim(x_1)$.

Assume that the DAE (2) is solvable and that E, F are $n \times n$. Differentiating the equation (2) j times gives the system of equations

$$\begin{bmatrix} \mathscr{E}_j & \mathscr{F}_j \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ x \end{bmatrix} = \mathbf{f}_j, \tag{4}$$

where

$$\mathscr{F}_j = \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(j)} \end{bmatrix}, \qquad \mathbf{f}_j = \begin{bmatrix} f \\ f' \\ \vdots \\ f^{(j)} \end{bmatrix}, \qquad \mathbf{x}_j = \begin{bmatrix} x' \\ \tilde{\mathbf{x}}_j \end{bmatrix}, \qquad \tilde{\mathbf{x}}_j = \begin{bmatrix} x'' \\ \vdots \\ x^{(j+1)} \end{bmatrix},$$

$$\mathscr{E}_j = \begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ E' + F & E & 0 & \ddots & \vdots \\ E'' + 2F' & 2E' + F & E & \ddots & 0 \\ * & * & * & \ddots & 0 \\ E^{(j)} + jF^{(j-1)} & * & * & * & E \end{bmatrix}.$$

From [5] we have

THEOREM 1. *Suppose that* (2) *is solvable on the interval* $\mathscr{I}$ *and that* E, F *are* $2n$ *times differentiable. Then*

$$\mathscr{E}_j \text{ has constant rank on } \mathscr{I} \text{ for } j = n+1, \tag{5a}$$

$$\mathscr{E}_j \text{ is 1-full with respect to } x' \text{ for } j = n+1, \tag{5b}$$

$$\begin{bmatrix} \mathscr{F}_j & \mathscr{E}_j \end{bmatrix} \text{ has full row rank for } 1 \leqslant j \leqslant n+1. \tag{5c}$$

If the coefficients E, F are infinitely differentiable, then Theorem 1 provides sufficient as well as necessary conditions for solvability. If (5c) holds, then the smallest value of j that satisfies the conditions (5a), (5b) of Theorem 1 is called the *index* ν of the DAE (2). We shall say that a submatrix of $[\mathscr{F}_j \ \ \mathscr{E}_j]$ satisfies (5) if it satisfies the conditions with the "for j" statements deleted.

3. VARIABILITY OF COMPLETIONS

Suppose for the remainder of this note that (2) is solvable and j is large enough that $\mathscr{E}_j, \mathscr{F}_j$ have the properties in (5). By Theorem 1 there exists nonsingular $\Theta(t)$ such that left multiplication on (4) yields

$$\begin{bmatrix} I & 0 \\ 0 & W \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x' \\ \tilde{\mathbf{x}}_j \end{bmatrix}+\begin{bmatrix} -G \\ Z \\ M \end{bmatrix}x=\begin{bmatrix} R_0 & \cdots & R_r \\ W_0 & \cdots & W_r \\ K_0 & \cdots & K_r \end{bmatrix}\mathbf{f}_j, \tag{6}$$

where W, M have full row rank. From (6) we get the *completion*

$$x' = G(t)x + \sum_{i=0}^{r} R_i(t) f^{(i)}(t) \tag{7}$$

of (2). The differential equation (7) is called a completion because it can be viewed as completing the vector field defined by (2). The bottom equation of (6),

$$Mx = \sum_{i=0}^{r} K_i(t) f^{(i)}(t), \tag{8}$$

characterizes the consistent initial conditions, or equivalently the solution manifold of (2) at time t [4]. To distinguish (7), which comes from the derivative array (4), from an arbitrary completion, we shall refer to (7) as a *natural completion*.

Since $\mathscr{E}_j$ has constant rank and is 1-full for all t, it is possible to choose $\Theta(t)$ to be as smooth as $\mathscr{E}_j$ is. The R_i, W_i, K_i are submatrices of Θ and are also as smooth as $\mathscr{E}_j$ is.

In most applications $\Theta(t)$ will represent the result of a numerical algorithm applied at time t. Thus it is important to examine to what extent that G, R_i are unique and how their smoothness is effected by the algorithm.

Assume that $\Theta(t)$ is smooth and gives (6). Let $\overline{\Theta}(t)$ be any other nonsingular matrix function (not necessarily smooth) which gives an equation in the form (6) but with coefficients

$$\begin{bmatrix} I & 0 \\ 0 & \overline{W} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -\overline{G} \\ \overline{Z} \\ \overline{M} \end{bmatrix}, \quad \begin{bmatrix} \overline{R}_0 & \cdots & \overline{R}_r \\ \overline{W}_0 & \cdots & \overline{W}_r \\ \overline{K}_0 & \cdots & \overline{K}_r \end{bmatrix}. \tag{9}$$

Let $\tilde{\Theta} = \overline{\Theta}\Theta^{-1}$, and define

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 & \tilde{\Theta}_2 & \tilde{\Theta}_3 \\ \tilde{\Theta}_4 & \tilde{\Theta}_5 & \tilde{\Theta}_6 \\ \tilde{\Theta}_7 & \tilde{\Theta}_8 & \tilde{\Theta}_9 \end{bmatrix}, \tag{10}$$

where the partition of (10) is conformal with (6), (9). Then

$$\tilde{\Theta}\begin{bmatrix} I & 0 & -G \\ 0 & W & Z \\ 0 & 0 & M \end{bmatrix} = \begin{bmatrix} I & 0 & -\overline{G} \\ 0 & \overline{W} & \overline{Z} \\ 0 & 0 & \overline{M} \end{bmatrix}. \tag{11}$$

From (11) we have $\tilde{\Theta}_5 W = \overline{W}$ and

$$\tilde{\Theta}_1 = I, \qquad \tilde{\Theta}_4 = 0, \qquad \tilde{\Theta}_7 = 0,$$

$$\tilde{\Theta}_2 W = 0, \qquad \tilde{\Theta}_8 W = 0. \tag{12}$$

Then W being full row rank and (12) imply that $\tilde{\Theta}_2 = \tilde{\Theta}_8 = 0$, so that

$$\tilde{\Theta} = \begin{bmatrix} I & 0 & \tilde{\Theta}_3 \\ 0 & \tilde{\Theta}_5 & \tilde{\Theta}_6 \\ 0 & 0 & \tilde{\Theta}_9 \end{bmatrix}. \tag{13}$$

Applying (13) now gives

PROPOSITION 1. *Suppose that* (2) *is solvable. If E is nonsingular, then* (7) *is unique and $r = 0$. If E is singular, then all possible natural completions are given by*

$$x' = \left(G - \tilde{\Theta}_3 M\right)x + \sum_{i=0}^{r} R_i f^{(i)} + \tilde{\Theta}_3 \sum_{i=0}^{r} K_i f^{(i)}. \tag{14}$$

Here $\tilde{\Theta}_3$ is an arbitrary $n \times \mathrm{rank}(M)$ matrix valued function, and G, M, R_i, K_i are from (6).

Of course we need $\tilde{\Theta}_3$ to be smooth if the completion given by (14) is to be smooth. If (2) is *totally singular*, that is, there is only one solution for each f, then (14) says that $G - \tilde{\Theta}_3 M$ is arbitrary, since M will be nonsingular.

Before examining the smoothness and uniqueness of G, R_i further, it is helpful to consider the special case when E, F are constant matrices. For this illustration, we may assume without loss of generality that

$$E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \qquad F = \begin{bmatrix} -F_1 & 0 \\ 0 & I \end{bmatrix}, \tag{15}$$

where $N^{\nu} = 0$, $N^{\nu-1} \neq 0$. Then Θ can be taken constant. There exist row operations on (4) which can be made to yield

$$G = \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M = [0 \quad I]. \tag{16}$$

Thus from (14) and (16) we get that the homogeneous part of any constant coefficient completion would be

$$\bar{G} = \begin{bmatrix} F_1 & \Phi_1 \\ 0 & \Phi_2 \end{bmatrix}, \tag{17}$$

where Φ_1, Φ_2 are arbitrary constant matrices.

In [8] we observed that from the derivative array (4) it is possible to get different spectral behavior from different completions. (See also [10].) Equation (17) shows much more.

PROPOSITION 2. *Suppose that E, F are constant matrices and the pencil E, F is equivalent to* (15). *Let F_1 have eigenvalues $\sigma(F_1)$. Suppose F_1 is $p \times p$. Let σ_2 be any set of $n - p$ numbers (not necessarily distinct). Then there is a completion computable from the derivative array* (4) *using elementary row operations such that $\sigma(G) = \sigma(F_1) \cup \sigma_2$.*

In practice the completion is usually computed numerically at times t_n with algorithms where the choice of pivot columns and rows may vary with t. One might expect that either there would often be cases where the coefficients are not smooth, or ensuring that the algorithm produces estimates of smooth coefficients would be difficult. Yet, numerical experiments to date have not encountered this difficulty [2, 6].

Part of the reason that discontinuity in G has not been observed is that Gx is unique if x is a consistent value for (2). However, there is another reason. We now show that, somewhat surprisingly, the smoothness is often already guaranteed by many existing algorithms.

Since $\mathscr{E}_j$ is both row and column rank deficient, it is usually best to use orthogonal matrix operations at first and then only use elementary row, or other nonorthogonal, operations after the full rank parts are safely determined. Such an approach usually consists of two steps. First orthogonal operations are performed on (4) to obtain

$$\begin{bmatrix} V & \Omega \\ 0 & \tilde{W} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -\tilde{G} \\ H \\ \tilde{M} \end{bmatrix}, \quad \begin{bmatrix} \tilde{R}_0 & \cdots & \tilde{R}_r \\ \tilde{W}_0 & \cdots & \tilde{W}_r \\ \tilde{K}_0 & \cdots & \tilde{K}_r \end{bmatrix}, \tag{18}$$

where V is nonsingular and $\tilde{W}$ is full row rank. Then elementary row operations equivalent to inverting V and using $\tilde{W}$ to eliminate Ω are performed to produce (6). Suppose that $\Theta, \overline{\Theta}$ are two orthogonal matrices with Θ producing (18) and $\overline{\Theta}$ producing

$$\begin{bmatrix} \overline{V} & \overline{\Omega} \\ 0 & \overline{W} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -\overline{G} \\ \overline{H} \\ \overline{M} \end{bmatrix}, \quad \begin{bmatrix} \overline{R}_0 & \cdots & \overline{R}_r \\ \overline{W}_0 & \cdots & \overline{W}_r \\ \overline{K}_0 & \cdots & \overline{K}_r \end{bmatrix} \tag{19}$$

with $\overline{V}$ nonsingular and $\overline{W}$ full row rank. Again let $\tilde{\Theta} = \overline{\Theta}\Theta^{-1}$ and use the notation of (10). Then

$$\tilde{\Theta}\begin{bmatrix} V & \Omega \\ 0 & \tilde{W} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \overline{V} & \overline{\Omega} \\ 0 & \overline{W} \\ 0 & 0 \end{bmatrix}, \tag{20}$$

so that $\tilde{\Theta}_4 V = 0$, $\tilde{\Theta}_7 V = 0$, and

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 & \tilde{\Theta}_2 & \tilde{\Theta}_3 \\ 0 & \tilde{\Theta}_5 & \tilde{\Theta}_6 \\ 0 & \tilde{\Theta}_8 & \tilde{\Theta}_9 \end{bmatrix}.$$

But $\tilde{\Theta}_1$ is orthogonal, since $\tilde{\Theta}$ is. The orthogonality of $\tilde{\Theta}$ then implies

$\tilde{\Theta}_2 = \tilde{\Theta}_3 = 0$ and

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 & 0 & 0 \\ 0 & \tilde{\Theta}_5 & \tilde{\Theta}_6 \\ 0 & \tilde{\Theta}_8 & \tilde{\Theta}_9 \end{bmatrix}.$$

Then (20) gives that $\tilde{\Theta}_8 \tilde{W} = 0$ and hence $\tilde{\Theta}_8 = 0$, since $\tilde{W}$ has full row rank. But then $\tilde{\Theta}_9$ is orthogonal and $\tilde{\Theta}_6 = 0$. We have finally that

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 & 0 & 0 \\ 0 & \tilde{\Theta}_5 & 0 \\ 0 & 0 & \tilde{\Theta}_9 \end{bmatrix}. \tag{21}$$

It follows from the 1-fullness of $\mathscr{E}_j$ that

$$\Omega = C\tilde{W} \tag{22}$$

for some matrix C which is as smooth as $\Omega, \tilde{W}$. Using the form (21) for $\tilde{\Theta}$, we have that (18) and (19) are related by

$$\tilde{\Theta}_1 V = \overline{V}, \qquad \tilde{\Theta}_1 \tilde{G} = \overline{G}, \qquad \tilde{\Theta}_1 \Omega = \overline{\Omega}, \qquad \tilde{\Theta}_1 \tilde{R}_i = \overline{R}_i, \tag{23a}$$

$$\tilde{\Theta}_5 \tilde{W} = \overline{W}, \qquad \tilde{\Theta}_5 H = \overline{H}, \qquad \tilde{\Theta}_5 \tilde{W}_i = \overline{W}_i, \tag{23b}$$

$$\tilde{\Theta}_9 \tilde{M} = \overline{M}, \qquad \tilde{\Theta}_9 \tilde{K}_i = \overline{K}_i. \tag{23c}$$

From (22) the completion computed from (18) is

$$x' = (V^{-1}\tilde{G} + V^{-1}CH)x + \sum_{i=0}^{r} (V^{-1}\tilde{R}_i - V^{-1}C\tilde{W}_i) f^{(i)}. \tag{24}$$

But $\overline{\Omega} = \tilde{\Theta}_1 \Omega = \tilde{\Theta}_1 C\tilde{W} = \tilde{\Theta}_1 C \tilde{\Theta}_5^{-1} \overline{W}$. The completion from (19) is then

$$x' = \left(\overline{V}^{-1}\overline{G} + \overline{V}^{-1}\left[\tilde{\Theta}_1 C \tilde{\Theta}_5^{-1}\right]\overline{H}\right)x + \sum \left(\overline{V}^{-1}\overline{R}_i - \overline{V}^{-1}\left[\tilde{\Theta}_1 C \tilde{\Theta}_5^{-1}\right]\overline{W}_i\right) f^{(i)}. \tag{25}$$

However, using (23) we compute that

$$\bar{V}^{-1}\bar{G}+\bar{V}^{-1}[\tilde{\Theta}_1 C\tilde{\Theta}_5^{-1}]\bar{H}=[\tilde{\Theta}_1 V]^{-1}\tilde{\Theta}_1\tilde{G}+[\tilde{\Theta}_1 V]^{-1}[\tilde{\Theta}_1 C\tilde{\Theta}_5^{-1}]\tilde{\Theta}_5 H$$
$$=V^{-1}\tilde{G}+V^{-1}CH$$

and

$$\bar{V}^{-1}\bar{R}_i+\bar{V}^{-1}[\tilde{\Theta}_1 C\tilde{\Theta}_5^{-1}]\bar{W}_i=[\tilde{\Theta}_1 V]^{-1}\tilde{\Theta}_1\tilde{R}_i+[\tilde{\Theta}_1 V]^{-1}[\tilde{\Theta}_1 C\tilde{\Theta}_5^{-1}]\tilde{\Theta}_5\tilde{W}_i$$
$$=V^{-1}\tilde{R}_i+V^{-1}C\tilde{W}_i.$$

Thus (24) and (25) are the same completions. We summarize this discussion in the next theorem.

THEOREM 2. *Assume that* (2) *is solvable and that* E, F *are smooth and* j *is large enough for the conditions* (5) *to hold. Suppose that the completion is computed pointwise by an algorithm which uses orthogonal operations to reduce* $\mathscr{E}_j$ *to the form* (18) *with* V *nonsingular and* $\tilde{W}$ *full row rank and then uses elementary row operations on the first two block rows to obtain* (6). *Then the completion is unique for a given* j *and is as smooth as* $\mathscr{E}_j, \mathscr{F}_j$ *are.*

Proof. We have shown the uniqueness in the preceding discussion. We have only to show the smoothness. Since $\mathscr{E}_j$ is 1-full with respect to x', the first n columns are linearly independent for all t. Thus there exists orthogonal Θ_1, as smooth as $\mathscr{E}_j$, such that

$$\Theta_1\mathscr{E}_j=\begin{bmatrix} V & \Omega \\ 0 & X \end{bmatrix}$$

and V is nonsingular. Now X will have constant rank, since $\mathscr{E}_j$ does. Thus there is an orthogonal Θ_2, again as smooth as $\mathscr{E}_j$, such that

$$\Theta_2 X=\begin{bmatrix} W \\ 0 \end{bmatrix},$$

where W has full row rank. ■

An alternative way to express Theorem 2 is the following. Let $A^{+}c$ denote any least squares solution of $Au=c$, and let $A^{\dagger}c$ denote the minimum norm least squares solution. Alternatively one may think of $A^{+}, A^{\dagger}$ as generalized inverses. Let π denote the first n rows of a matrix. Note that $\pi(CD) = \pi(C)D$.

PROPOSITION 3. *The completions derived in Theorem* 2 *may be written*

$$x' = \pi(\mathcal{E}_j^{+}\mathcal{F}_j)x + \pi(\mathcal{E}_j^{+})\mathbf{f}_j, \tag{26}$$

where $\pi(\mathcal{E}_j^{+}\mathcal{F}_j) = \pi(\mathcal{E}_j^{\dagger}\mathcal{F}_j)$, $\pi(\mathcal{E}_j^{+}) = \pi(\mathcal{E}_j^{\dagger})$.

We shall call (26) a *least squares completion*, or *ls completion* for short.

In working with a particular system the value of j may be larger then necessary. Also, it sometimes reduces the computational cost to use a smaller set of equations if possible. In either case it is important to know what effect this has on the completion. In general, two different ls completions need not be identical.

EXAMPLE 1. Consider the following index one linear time invariant DAE:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x' + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = f. \tag{27}$$

Taking $j = 1$, we have

$$[\mathcal{E}_1 | \mathcal{F}_1] = \left[\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \end{array}\right]. \tag{28}$$

The ls completion from (28) is

$$x' = -\begin{bmatrix} 1 & 1.5 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} f + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} f'. \tag{29}$$

However, if we take the submatrix of (28) consisting of the first, third, and fourth rows, we get

$$\left[\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \end{array}\right], \tag{30}$$

which is still 1-full with respect to x'. The ls completion computed from (30) is

$$x' = -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} f + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} f', \tag{31}$$

which is different from (29). The second, third, and fourth rows of (28) also give a derivative array which is 1-full with respect to x'. The ls completion from this subarray is

$$x' = -\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} f + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} f'.$$

In Example 1 the submatrix (30) was large enough to be 1-full with respect to x', but it was not large enough to determine the solution manifold (8) for (27). If we only consider derivative arrays that are large enough to determine not only x' but also the constraints, then we do have uniqueness of ls completions.

PROPOSITION 4. *Suppose that j is large enough so that $[\mathscr{E}_j \;\; \mathscr{F}_j]$ has full row rank and is 1-full with respect to x', and $\mathscr{E}_j$ has constant rank. Let $[\overline{\mathscr{E}}_j \;\; \overline{\mathscr{F}}_j]$ be any submatrix consisting of rows of $[\mathscr{E}_j \;\; \mathscr{F}_j]$ which has these same three properties and* $\operatorname{nullity}(\overline{\mathscr{E}}_j) = \operatorname{nullity}(\mathscr{E}_j)$. *Then the ls completions computed from $[\mathscr{E}_j \;\; \mathscr{F}_j]$ and $[\overline{\mathscr{E}}_j \;\; \overline{\mathscr{F}}_j]$ are the same.*

Proof. Let P be a permutation matrix such that

$$P[\mathscr{E}_j \quad \mathscr{F}_j] = \left[\begin{array}{c|c} \overline{\mathscr{E}}_j & \overline{\mathscr{F}}_j \\ X_1 & X_2 \end{array}\right]. \tag{32}$$

By assumption we may perform orthogonal operations which only change the first block row of (32) to get

$$\left[\begin{array}{cc|c} V & \Omega & -G \\ 0 & W & H \\ 0 & 0 & M \\ X_{11} & X_{12} & X_2 \end{array}\right],$$

where V is nonsingular, W has full row rank, and $\mathrm{rank}(M) = \mathrm{nullity}(\mathscr{E}_j) = \mathrm{nullity}(\bar{\mathscr{E}}_j)$. But then

$$\begin{bmatrix} V & \Omega \\ 0 & W \\ X_{11} & X_{12} \end{bmatrix}$$

has to have full row rank. In order to get the respective ls completions, we need only do elementary row operations on the full row rank matrices

$$\left[\begin{array}{cc|c|ccc} V & \Omega & -G & \tilde{R}_0 & \cdots & \tilde{R}_r \\ 0 & W & H & \tilde{W}_0 & \cdots & \tilde{W}_r \end{array}\right] \tag{33}$$

and

$$\left[\begin{array}{cc|c|ccc} V & \Omega & -G & \tilde{R}_0 & \cdots & \tilde{R}_r \\ 0 & W & H & \tilde{W}_0 & \cdots & \tilde{W}_r \\ X_{11} & X_{12} & X_2 & \tilde{Z}_0 & \cdots & \tilde{Z}_r \end{array}\right]. \tag{34}$$

But (33) and (34) both yield (24). ■

4. CONCLUSION

Differentiation of differential algebraic equations is used for many purposes, ranging from index reduction to control techniques such as system inversion. General numerical methods, which can be used when other, more efficient methods such as implicit Runge-Kutta (IRK) or backward differentiation (BDF) fail, have been proposed based on derivative arrays. Most of these approaches consist of determining all or part of a completion of the original vector field defined by the DAE. In more complex problems these completions would probably be determined numerically. That the differentiated equations do not uniquely determine a completion is known [7, 10]. However, we have shown that the introduced dynamics are essentially arbitrary off the solution manifold if no restrictions are placed on the numerical method computing the completion, and the coefficients do not even have to be smooth if different values of t would lead, for example, to changes in choice of pivots during Gaussian elimination.

On the other hand, because of the rank deficiency of the derivative arrays, most problems examined to date have used orthogonal operations at least until the rank is determined. Let us call such an algorithm *standard*. We have shown that standard algorithms always generate a smooth completion for a given derivative array satisfying the rank properties (5). This completion, called the ls completion, is the same for all standard algorithms applied to a given derivative array. Finally, if we limit ourselves to derivative arrays which are large enough to also determine the solution manifold, then the ls completion of a given DAE (2) is independent of both the particular array and which standard algorithm is used.

REFERENCES

1 K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Elsevier, 1989.

2 S. L. Campbell, The numerical solution of higher index linear time varying singular systems of differential equations, *SIAM J. Sci. Statist. Comput.* 6:334–348 (1985).

3 S. L. Campbell, Rank deficient least squares and the numerical solution of linear singular implicit systems of differential equations, in *Linear Algebra and Its Role in Systems Theory*, Contemp. Math. 47, 1985, pp. 51–63.

4 S. L. Campbell, Consistent initial conditions for linear time varying singular systems, in *Frequency Domain and State Space Methods* (C. I. Byrnes and A. Lindquist, Eds.), Elsevier, 1986, pp. 313–318.

5 S. L. Campbell, A general form for solvable linear varying singular systems of differential equations, *SIAM J. Math. Anal.* 18:1101–1115 (1987).

6 S. L. Campbell, A computational method for general higher index singular systems of differential equations, in *1989 IMACS Transactions Scientific Computing*, Vol. 1.2, 1989, pp. 555–560.

7 S. L. Campbell and B. Leimkuhler, Differentiation of constraints in differential algebraic equations, Preprint, 1989.

8 S. L. Campbell and W. J. Terrell, Observability for linear time varying descriptor systems, *SIAM J. Matrix Anal. Appl.*, to appear.

9 K. D. Clark and L. Petzold, Numerical Solution of Boundary Value Problems in Differential/Algebraic Equations, *Siam J. Sci. Statist. Comput.* 10:915–936 (1989).

10 T. Mrziglod, Zur Theorie und numerischen Realisierung von Lösungmethoden bei Differentialgleichungen mit angekoppelten algebraischen Gleichungen, Diplomarbeit, Mathematischen Inst., Univ. zu Köln, 1987.

11 L. M. Silverman and H. E. Meadows, Controllability and observability in time-variable linear systems, *SIAM J. Control* 5:64–73 (1967).

12 L. M. Silverman, Inversion of multivariable systems, *IEEE Trans. Automat. Control* AC-14:270–276 (1969).

Received 14 May 1990; final manuscript accepted 28 November 1990